MAA AMC
American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

25th Annual

AMC 10 A

Wednesday, November 8, 2023

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 10 A were prepared by the
MAA AMC 10/12 Editorial Board under the direction of
Gary Gordon and Carl Yerger, co-Editors-in-Chief.

1. Cities A and B are 45 miles apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 miles per hour. Leaving at the same time, Beth bikes toward A at 12 miles per hour. How many miles from City A will they be when they meet?

   (A) 20 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

   Answer (E): Let t be the number of hours they ride before meeting. Then $18t + 12t = 45$. This gives $t = 1.5$, so they each ride for 1.5 hours. Then Alicia has biked $18 \cdot 1.5 = 27$ miles, so their meeting point is 27 miles from A.

2. The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

   (A) $1\dfrac{4}{5}$ **(B)** 2 **(C)** $2\dfrac{2}{5}$ **(D)** 3 **(E)** $3\dfrac{3}{5}$

   Answer (A): Let p be the weight of a large pizza and q be the weight of a cup of orange slices, in pounds. Then
$$\frac{1}{3}p + \frac{7}{2}q = \frac{3}{4}p + \frac{1}{2}q,$$
   so $3q = \left(\frac{3}{4} - \frac{1}{3}\right) p$ and $p = \frac{36}{5}q$. Because a cup of orange slices weighs $\frac{1}{4}$ of a pound, the large pizza weighs
$$\frac{36}{5} \cdot \frac{1}{4} = \frac{9}{5} = 1\frac{4}{5} \text{ pounds.}$$

3. How many positive perfect squares less than 2023 are divisible by 5 ?

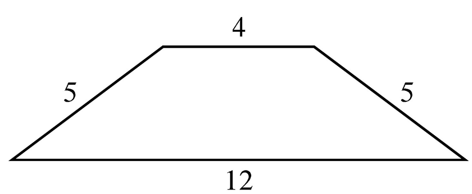
   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

   Answer (A): Because 5 is prime, a perfect square is divisible by 5 if and only if its square root is divisible by 5. Note that $40^2 = 1600 < 2023 < 2025 = 45^2$, so 5^2, 10^2, 15^2, 20^2, 25^2, 30^2, 35^2, and 40^2 are the only positive perfect squares satisfying the given condition. There are 8 such numbers.

4. A quadrilateral has all integer side lengths, a perimeter of 26, and one side of length 4. What is the greatest possible length of one side of this quadrilateral?

   (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

   Answer (D): The longest side must have length less than the sum of the other three side lengths. Because the sum of all the side lengths is 26, this implies that the longest side length must be less than 13, and because all the side lengths are positive integers, it must be less than or equal to 12. Such a quadrilateral could have side lengths 12, 4, 5, and 5, for example, as shown in the figure.



5. How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

   (A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

   Answer (E): Because $8 = 2^3$ and $15 = 3 \cdot 5$, the prime factorization of the given number is $2^{15} \cdot 3^5 \cdot 5^{15}$, which is $3^5 \cdot 10^{15} = 243 \cdot 10^{15}$. The base-ten representation is 243 followed by 15 zeros, so it has 18 digits in all.

6. An integer is assigned to each vertex of a cube. The value of an edge is defined to be the sum of the values of the two vertices it touches, and the value of a face is defined to be the sum of the values of the four edges surrounding it. The value of the cube is defined as the sum of the values of its six faces. Suppose the sum of the integers assigned to the vertices is 21. What is the value of the cube?

   (A) 42 **(B)** 63 **(C)** 84 **(D)** 126 **(E)** 252

   Answer (D): Each vertex of the cube touches 3 edges, so the sum of the values of the 12 edges is 3 times the sum of the values of the 8 vertices. Each edge touches 2 faces, so the sum of the values of the faces is twice the sum of the values of the edges. Thus the value of the cube is $2 \cdot 3 \cdot 21 = 126$.

   For example, assigning 21 to one vertex and 0 to the other seven vertices will result in the given values. Specifically, there will be three edges with value 21 and nine edges with value 0. This means that there will be three faces with value 42 and three faces with value 0.

7. Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal 3 ?

   (A) $\dfrac{2}{9}$ **(B)** $\dfrac{49}{216}$ **(C)** $\dfrac{25}{108}$ **(D)** $\dfrac{17}{72}$ **(E)** $\dfrac{13}{54}$

   Answer (B): Janet rolls a total of 3 if she starts by rolling 111, 12, 21, or 3. These initial sequences of rolls occur with probability $\frac{1}{6^3}$, $\frac{1}{6^2}$, $\frac{1}{6^2}$, and $\frac{1}{6}$, respectively. Adding these probabilities gives

$$\frac{1}{6^3} + \frac{1}{6^2} + \frac{1}{6^2} + \frac{1}{6} = \frac{49}{216}.$$

8. Barb the baker has developed a new temperature scale for her bakery called the Breadus scale, which is a linear function of the Fahrenheit scale. Bread rises at 110 degrees Fahrenheit, which is 0 degrees on the Breadus scale. Bread is baked at 350 degrees Fahrenheit, which is 100 degrees on the Breadus scale. Bread is done when its internal temperature is 200 degrees Fahrenheit. What is this in degrees on the Breadus scale?

   (A) 33 **(B)** 34.5 **(C)** 36 **(D)** 37.5 **(E)** 39

   Answer (D): The difference between the bread rising temperature and the bread baking temperature in degrees Fahrenheit is $350 - 110 = 240$. The Fahrenheit temperature of bread when it is done baking is 200 degrees, which is $200 - 110 = 90$ degrees above that of bread rising, so is $\frac{90}{240} = \frac{3}{8}$ of the way from 110 to 350. Therefore on the Breadus scale this is $\frac{3}{8}$ of the way from 0 to 100. This value is

$$\frac{3}{8} \cdot (100 - 0) = \frac{300}{8} = 37.5.$$

OR

The graph of the number of degrees Breadus (y-axis) as a function of the number of degrees Fahrenheit (x-axis) is a line that passes through the points $(110, 0)$ and $(350, 100)$. The slope of this line is

$$\frac{100 - 0}{350 - 110} = \frac{5}{12}.$$

The equation of the line is therefore $y = \frac{5}{12}(x - 110)$, and the value of y when $x = 200$ is $\frac{5}{12} \cdot 90 = 37.5$.

OR

The internal temperature of bread in degrees Fahrenheit when it is done baking is a weighted average (a convex combination) of the extremes of 110 and 350, so $110x + 350(1-x) = 200$ for some value x. Solving for x gives $x = \frac{5}{8}$. Substituting this in the convex combination for internal temperature of bread in degrees Breadus when it is done baking gives

$$0x + 100(1 - x) = 0 \cdot \frac{5}{8} + 100 \cdot \frac{3}{8} = 37.5.$$

9. A digital display shows the current date as an 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as 20230428. For how many dates in 2023 does each digit appear an even number of times in the 8-digit display for that date?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (E): Because the first four digits of the display must be 2023, the last four digits must contain one 0, one 3, and two of the same digit. These digits cannot contain three 0s or three 3s, so the only possible repeated digits among them are 1 and 2.

If the repeated digit is 1, the possible dates are 01/13, 01/31, 03/11, 10/13, 10/31, 11/03, and 11/30. If the repeated digit is 2, the possible dates are 02/23 and 03/22. In all there are 9 dates in 2023 for which each digit appear an even number of times.

10. Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D): Suppose Maureen has taken n quizzes so far, with mean m. Then

$$\frac{mn + 11}{n + 1} = m + 1 \qquad \text{and} \qquad \frac{mn + 3 \cdot 11}{n + 3} = m + 2.$$

These two equations simplify to $11 = m + n + 1$ and $33 = 3m + 2n + 6$, respectively. Subtracting twice the first equation from the second equation yields $11 = m + 4$, so $m = 7$. Then $n = 3$, so

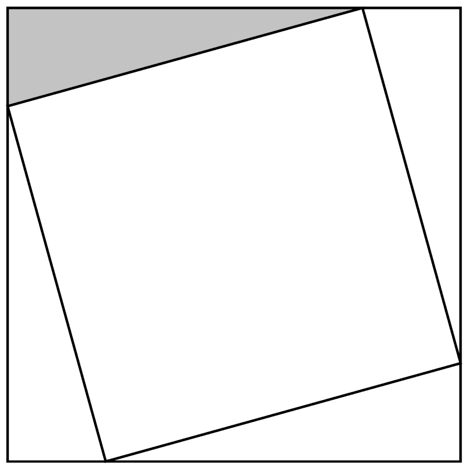
Maureen has taken 3 quizzes so far, and their scores could, for example, be 6, 7, and 8. Then one score of 11 increases the mean to

$$\frac{6+7+8+11}{4} = \frac{32}{4} = 8,$$

and three scores of 11 increase the mean to

$$\frac{6+7+8+11+11+11}{6} = \frac{54}{6} = 9.$$

11. A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $2-\sqrt{3}$ **(D)** $\sqrt{3}-\sqrt{2}$ **(E)** $\sqrt{2}-1$

Answer (C): Let x be the length of the shorter leg of the right triangle. Then the longer leg has length $\sqrt{3}-x$. The hypotenuse has length $\sqrt{2}$. By the Pythagorean Theorem,

$$x^2 + \left(\sqrt{3}-x\right)^2 = 2.$$

This equation simplifies to $2x^2 - 2\sqrt{3}x + 1 = 0$, and the Quadratic Formula gives the solutions $\frac{1}{2}\left(\sqrt{3} \pm 1\right)$. The requested ratio is

$$\frac{\frac{1}{2}\left(\sqrt{3}-1\right)}{\frac{1}{2}\left(\sqrt{3}+1\right)} = 2-\sqrt{3}.$$

12. How many three-digit positive integers N satisfy the following properties?

- The number N is divisible by 7.
- The number formed by reversing the digits of N is divisible by 5.

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

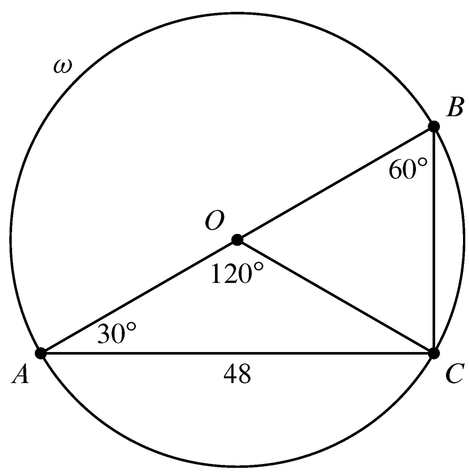
Answer (B): Let M be the number formed by reversing the digits of N. Because M is divisible by 5, the last digit of M must be 0 or 5. However, the last digit of M cannot be 0, so it must be 5. It follows that N is a number between 500 and 599.

The least multiple of 7 in this range is $504 = 7 \cdot 72$, while the greatest multiple of 7 in this range is $595 = 7 \cdot 85$. Thus N is one of the numbers $7 \cdot 72, 7 \cdot 73, 7 \cdot 74, \ldots,$ or $7 \cdot 85$. There are $85 - 72 + 1 = 14$ numbers in this list.

13. Abdul and Chiang are standing 48 feet apart in a field. Bharat is standing in the same field as far from Abdul as possible so that the angle formed by his lines of sight to Abdul and Chiang measures $60°$. What is the square of the distance (in feet) between Abdul and Bharat?

(A) 1728 **(B)** 2601 **(C)** 3072 **(D)** 4608 **(E)** 6912

Answer (C): In the diagram below—an overhead view—points A, B, and C represent the locations of Abdul, Bharat, and Chiang, respectively. Let ω be the circle containing A, B, and C, and let O be center of ω. Because $\angle ABC = 60°$, it follows that $\angle AOC = 120°$.



Because B is as far as possible from A, it follows that B must be at the other end of the diameter of ω containing A, as shown.

It remains to calculate AB. Because $\triangle ABC$ is a $30-60-90°$ right triangle and $AC = 48$, it follows that $AB = \frac{96}{\sqrt{3}}$. Then

$$AB^2 = \frac{96^2}{3} = 3072.$$

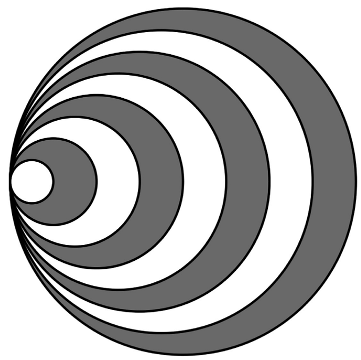
14. A number is chosen at random from among the first 100 positive integers, and a positive integer divisor of that number is then chosen at random. What is the probability that the chosen divisor is divisible by 11 ?

(A) $\frac{4}{100}$ **(B)** $\frac{9}{200}$ **(C)** $\frac{1}{20}$ **(D)** $\frac{11}{200}$ **(E)** $\frac{3}{50}$

Answer (B): There are 9 multiples of 11 among the first 100 positive integers, and the probability of choosing any particular one of them is $\frac{1}{100}$. For each of these multiples of 11, the probability that a randomly chosen divisor is a multiple of 11 is $\frac{1}{2}$ because each divisor that is a multiple of 11 can be paired with a divisor that is not a multiple of 11. Therefore the requested probability is

$$\frac{9}{100} \cdot \frac{1}{2} = \frac{9}{200}.$$

15. An even number of circles are nested, starting with a radius of 1 and increasing by 1 each time, all sharing a common point. The region between every other circle is shaded, starting with the region inside the circle of radius 2 but outside the circle of radius 1. An example showing 8 circles is displayed below. What is the least number of circles needed to make the total shaded area at least 2023π ?



(A) 46 **(B)** 48 **(C)** 56 **(D)** 60 **(E)** 64

Answer (E): Suppose there are $2n$ circles. The shaded area is then

$$\begin{aligned} A &= \pi\left(2^2 - 1^2\right) + \pi\left(4^2 - 3^2\right) + \cdots + \pi\left((2n)^2 - (2n-1)^2\right) \\ &= 3\pi + 7\pi + \cdots + (4n-1)\pi \\ &= \pi \sum_{k=1}^{n} (4k-1) \\ &= \frac{\pi(3 + 4n - 1)\,n}{2} \\ &= n(2n+1)\pi. \end{aligned}$$

To find the least value of n such that $A \geq 2023\pi$, note that $n \approx \sqrt{\frac{2023}{2}} \approx 32$. Because $31(2 \cdot 31 + 1) = 1953$ and $32(2 \cdot 32 + 1) = 2080$, the required value is $n = 32$, and there are $2 \cdot 32 = 64$ circles.

16. In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was 40% more than the number of games won by right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

(A) 15 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 66

Answer (B): Let ℓ denote the number of left-handed players, and let r denote the number of right-handed players. There were $\frac{\ell(\ell-1)}{2}$ games between two left-handed players, $\frac{r(r-1)}{2}$ games between two right-handed players, and $\ell \cdot r$ games involving one left-handed and one right-handed player. Let n denote the number of games won by right-handed players against left-handed players. Then right-handed players won a total of $\frac{r(r-1)}{2} + n$ games, and left-handed players won $\frac{\ell(\ell-1)}{2} + (\ell \cdot r - n)$ games. Because left-handed players won 40% more games than right-handed players,

$$\frac{\ell(\ell-1)}{2} + \ell \cdot r - n = 1.4\left(\frac{r(r-1)}{2} + n\right).$$

Using $r = 2\ell$ and simplifying this equation gives $\ell^2 - 3\ell + 8n = 0$. If $n \geq 1$, there are no real solutions for this quadratic equation. Therefore $n = 0$, giving $\ell = 3$ and $r = 6$. The total number of games played is

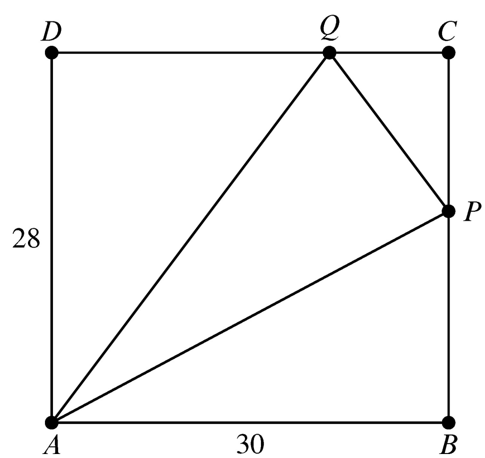
$$\frac{\ell(\ell-1)}{2} + \frac{r(r-1)}{2} + \ell \cdot r = 3 + 15 + 18 = 36.$$

Note that the $r = 6$ right-handed players won a total of 15 games (all among themselves), and the $\ell = 3$ left-handed players won a total of 21 games (3 games among themselves and all 18 games against right-handed players), and 21 is indeed 40% more than 15.

17. Let $ABCD$ be a rectangle with $AB = 30$ and $BC = 28$. Points P and Q lie on $\overline{BC}$ and $\overline{CD}$, respectively, so that all sides of $\triangle ABP$, $\triangle PCQ$, and $\triangle QDA$ have integer lengths. What is the perimeter of $\triangle APQ$?

(A) 84 **(B)** 86 **(C)** 88 **(D)** 90 **(E)** 92

Answer (A): The figure below shows the given information.



The Pythagorean triples with smallest total lengths are $3-4-5$, $5-12-13$, $7-24-25$, $8-15-17$, and multiples thereof. Note that $AD = 28 = 4 \cdot 7$. Then triangle $\triangle ADQ$ must be a multiple of $3-4-5$ (with multiplier 7) or a multiple of $7-24-25$ (with multiplier 4). But the latter option implies $DQ = 4 \cdot 24 > 30$, so the first option is correct. This gives $AQ = 5 \cdot 7 = 35$.

Because $DQ = 3 \cdot 7 = 21$, it follows that $QC = 30 - 21 = 9$. Thus triangle $\triangle QCP$ is also a multiple of $3-4-5$ (with multiplier 3). This gives $PC = 4 \cdot 3 = 12$ and $PQ = 5 \cdot 3 = 15$.

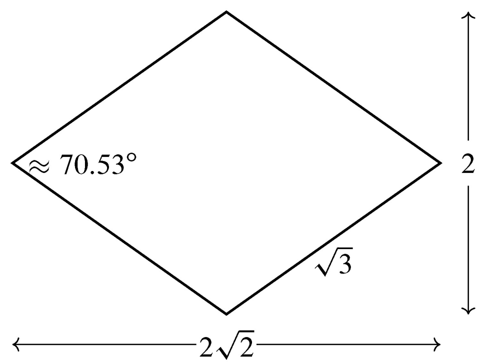
Finally, because $PC = 12$, it follows that $PB = BC - PC = 28 - 12 = 16$. Then triangle $\triangle ABQ$ must be a multiple of $8-15-17$ (with multiplier 2), so $AP = 17 \cdot 2 = 34$. The perimeter of $\triangle APQ$ is $34 + 15 + 35 = 84$.

18. A *rhombic dodecahedron* is a convex polyhedron where each of the 12 faces is a rhombus, and all of the faces are congruent to each other. The number of edges that meet at a vertex is either 3 or 4, depending on the vertex. What is the number of vertices at which exactly 3 edges meet?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (D): Let V, E, and F denote the number of vertices, edges, and faces of the rhombic dodecahedron, respectively. Thus $F = 12$. Each face has 4 edges, and each edge belongs to exactly 2 faces, so $E = \frac{4 \cdot 12}{2} = 24$. Euler's Formula, $V - E + F = 2$, gives $V = 14$. Let V_3 denote the number of vertices at which 3 edges meet, and let V_4 the number of vertices at which 4 edges meet. Thus $V_3 + V_4 = 14$. The quantity $3V_3 + 4V_4$ counts each edge twice, once for each of its endpoints, so $3V_3 + 4V_4 = 48$. Solving this system of equations yields $V_3 = 8$ and $V_4 = 6$.

Note: In order to form a rhombic dodecahedron, it is necessary that each rhombus has the dimensions shown below.



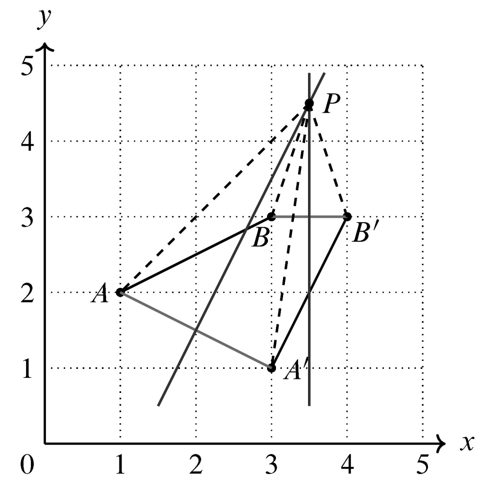
19. The line segment from $A(1,2)$ to $B(3,3)$ can be transformed to the line segment from $A'(3,1)$ to $B'(4,3)$, sending A to A' and B to B', by a rotation centered at the point $P(s,t)$. What is $|s-t|$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** 1

Answer (E): Because A and A' are the same distance from the center of rotation, $P(s,t)$ must lie on the perpendicular bisector of $\overline{AA'}$; similarly it must lie on the perpendicular bisector of $\overline{BB'}$. The line $\overline{AA'}$ passes through the midpoint $\left(2, \frac{3}{2}\right)$ and has slope $-\frac{3-1}{1-2} = 2$, so its equation is

$$y = 2(x-2) + \frac{3}{2}.$$

The line $\overline{BB'}$ is vertical and passes through $\left(\frac{7}{2}, 3\right)$, so its equation is $x = \frac{7}{2}$. Solving this system of equations gives $P(s,t) = \left(\frac{7}{2}, \frac{9}{2}\right)$, as shown in the figure. The requested absolute difference of coordinates is $\left|\frac{7}{2} - \frac{9}{2}\right| = 1$.

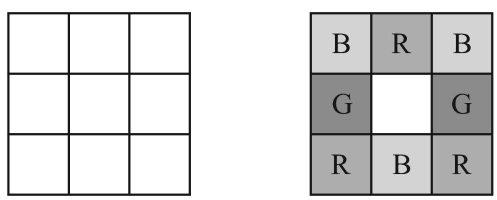


Note: As a check, note that the angle of rotation must be $\angle APA'$ and also $\angle BPB'$; it is about 37°. By the SSS Congruence Theorem, $\triangle APB \cong \triangle A'PB'$, so indeed

$$\angle APA' = \angle APB + \angle BPA' = \angle A'PB' + \angle BPA' = \angle BPB'.$$

Thus the rotation of this amount around $\left(\frac{7}{2}, \frac{9}{2}\right)$ is the required transformation.

20. Each square in a 3×3 grid of squares is colored red, white, blue, or green so that every 2×2 square contains one square of each color. One such coloring is shown on the right below. How many different colorings are possible?



(A) 24 **(B)** 48 **(C)** 60 **(D)** 72 **(E)** 96

Answer (D): Label the squares as shown.

a	b	c
d	e	f
g	h	i

There are four 2×2 squares that need to satisfy the condition: $abde$, $bcef$, $degh$, and $efhi$.

- Square $abde$: There are $4! = 24$ ways to color these squares using all four colors.
- Square $bcef$: Since squares b and e have already been colored, either c receives the same color as a and f receives the same color as d, or these are switched. This gives 2 choices at this stage. Using the numbers 1, 2, 3, and 4 to represent the colors, the two possibilities are pictured below.

1	2	1
3	4	3

1	2	3
3	4	1

- Remaining squares on the bottom row: Squares d and e have already been colored. If a and c have the same color, then there are two ways to color the three remaining squares:

1	2	1
3	4	3
1	2	1

1	2	1
3	4	3
2	1	2

On the other hand, if squares a and c receive different colors, then there is only one way to color the bottom row:

1	2	3
3	4	1
1	2	3

Thus the number of colorings is $24(2 + 1) = 72$.

21. Let $P(x)$ be the unique polynomial of minimal degree with the following properties:

- $P(x)$ has leading coefficient 1,
- 1 is a root of $P(x) - 1$,
- 2 is a root of $P(x - 2)$,
- 3 is a root of $P(3x)$, and
- 4 is a root of $4P(x)$.

The roots of $P(x)$ are integers, with one exception. The root that is not an integer can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 41 **(B)** 43 **(C)** 45 **(D)** 47 **(E)** 49

Answer (D): The given conditions imply that

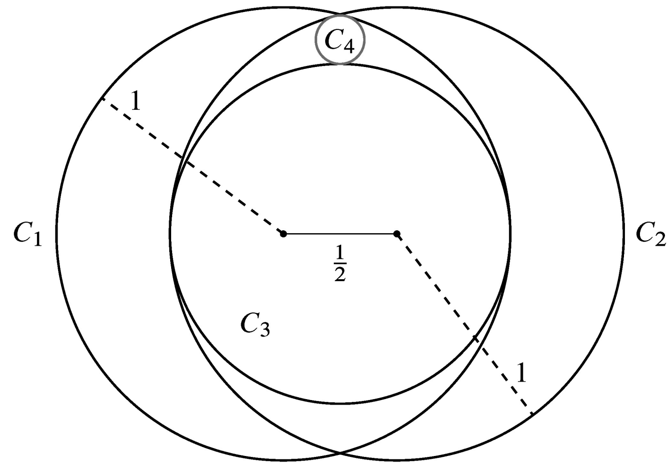
$$P(1) = 1, \quad P(2-2) = P(0) = 0, \quad P(3 \cdot 3) = P(9) = 0, \quad 4P(4) = P(4) = 0.$$

The polynomial $Q(x) = x(x - 4)(x - 9)$ has the required roots, but $Q(1) = 24 \neq 1$. Because the leading coefficient of $P(x)$ is 1, it follows that $P(x)$ must have an additional factor. To minimize the degree of $P(x)$, that factor must be linear, say $x - a$. Thus $P(x) = (x - a)Q(x)$, and

$$1 = P(1) = (1 - a) \cdot 1 \cdot (-3) \cdot (-8) = 24(1 - a).$$

Thus $a = \frac{23}{24}$, and the requested sum is $23 + 24 = 47$.

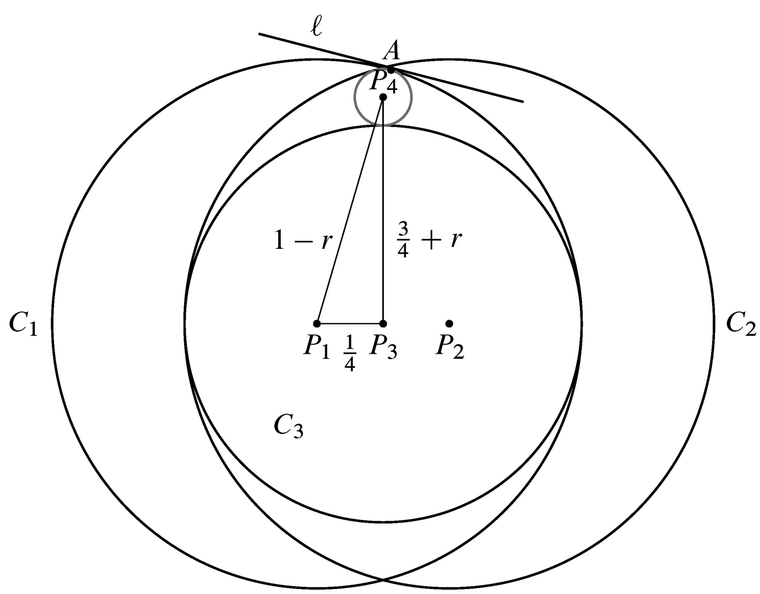
22. Circles C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4 ?



(A) $\dfrac{1}{14}$ **(B)** $\dfrac{1}{12}$ **(C)** $\dfrac{1}{10}$ **(D)** $\dfrac{3}{28}$ **(E)** $\dfrac{1}{9}$

Answer (D):

Let r be the radius of C_4. Let the point P_k be the center of C_k for $1 \le k \le 4$. Let A be the point of intersection of circles C_1 and C_4 and let ℓ be the tangent line to P_1 at A. By symmetry, P_3 is the midpoint of $\overline{P_1P_2}$, and $P_1P_3 = \frac{1}{4}$. Also by symmetry, P_4 lies on the perpendicular bisector of $\overline{P_1P_2}$. The radius $\overline{P_1A}$ is perpendicular to ℓ. Likewise the radius $\overline{P_4A}$ is perpendicular to ℓ, so P_1, P_4, and A all lie on the segment $\overline{P_1A}$ with P_4 between P_1 and A. Then $P_1P_4 = 1 - r$.



The radius of C_3 is $\frac{3}{4}$. Then $P_3P_4 = \frac{3}{4} + r$, and $\triangle P_1P_3P_4$ is a right triangle with $P_1P_3 = \frac{1}{4}$,

$P_3P_4 = \frac{3}{4} + r$, and $P_1P_4 = 1 - r$. By the Pythagorean Theorem,

$$\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4} + r\right)^2 = (1-r)^2.$$

Expanding gives

$$\frac{1}{16} + \frac{9}{16} + \frac{3r}{2} + r^2 = 1 - 2r + r^2,$$

which gives $r = \frac{3}{28}$.

23. If the positive integer c has positive integer divisors a and b with $c = ab$, then a and b are said to be *complementary* divisors of c. Suppose that N is a positive integer that has one complementary pair of divisors that differ by 20 and another pair of complementary divisors that differ by 23. What is the sum of the digits of N ?

(A) 9 **(B)** 13 **(C)** 15 **(D)** 17 **(E)** 19

Answer (C): Write the factorizations as $a(a+20) = b(b+23) = N$. Thus $0 < b < a$. Multiply through by 4 and complete the square to see that

$$(2a+20)^2 = (2b+23)^2 - 129.$$

Thus

$$(2b+23)^2 - (2a+20)^2 = (2b+2a+43)(2b-2a+3) = 129 = 3 \cdot 43.$$

The positive divisors of 129 are 1, 3, 43, and 129. The number $2b + 2a + 43$ is a divisor of 129 that is greater than 43, so

$$2b + 2a + 43 = 129$$

and

$$2b - 2a + 3 = 1.$$

This gives $a = 22$ and $b = 21$, so $a+20 = 42$, $b+23 = 44$, and hence $N = 22 \cdot 42 = 2^2 \cdot 3 \cdot 7 \cdot 11 = 21 \cdot 44 = 924$. The sum of its digits is $9 + 2 + 4 = 15$.

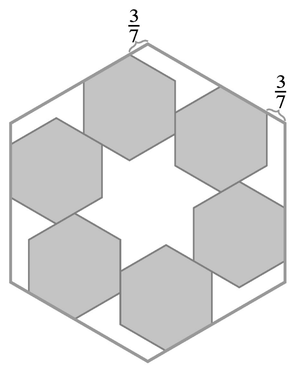
OR

As in the previous solution, if the factorizations are written as $a(a+20) = b(b+23) = N$, then $b < a$ and $a + 20 < b + 23$, from which $b < a < b + 3$.

Because a and b must be integers, a must equal $b+1$ or $b+2$. In the case when $a = b+1$, $a(a+20)$ becomes $(b+1)(b+21) = b^2 + 22b + 21$. Setting this equal to $b(b+23) = b^2 + 23b$ yields the solution $b = 21$, which corresponds to $N = 21 \cdot 44 = 22 \cdot 42 = 924$, and the sum of the digits of 924 is 15.

In the case when $a = b+2$, $a(a+20)$ becomes $(b+2)(b+22) = b^2 + 24b + 44$. Setting this equal to $b(b+23) = b^2 + 23b$ yields the only solution, $b = -44$. This does not meet the requirement that the complementary divisors be positive, although here as well $(-44) \cdot (-21) = (-42) \cdot (-21) = 924$.

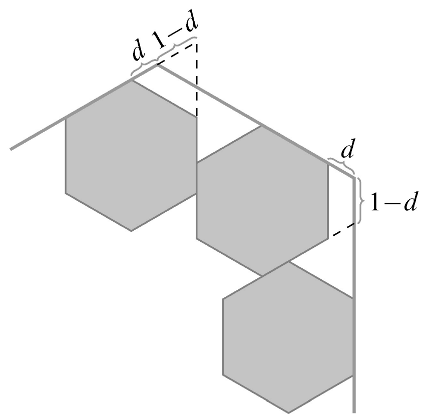
24. Six regular hexagonal blocks of side length 1 unit are arranged inside a regular hexagonal frame. Each block lies along an inside edge of the frame and is aligned with two other blocks, as shown in the figure below. The distance from any corner of the frame to the nearest vertex of a block is $\frac{3}{7}$ unit. What is the area of the region inside the frame not occupied by the blocks?



(A) $\dfrac{13\sqrt{3}}{3}$ **(B)** $\dfrac{216\sqrt{3}}{49}$ **(C)** $\dfrac{9\sqrt{3}}{2}$ **(D)** $\dfrac{14\sqrt{3}}{3}$ **(E)** $\dfrac{243\sqrt{3}}{49}$

Answer (C): The required area equals the difference between the area enclosed by the hexagonal frame and the area of the 6 hexagonal blocks. A block is composed of 6 unit equilateral triangles, each of area $\frac{\sqrt{3}}{4}$ square units, so the 6 blocks have a total area of $6 \cdot 6 \cdot \frac{\sqrt{3}}{4} = 9\sqrt{3}$ square units.

Let d represent the distance from a corner of the frame to the nearest block vertex. To determine the side length of the frame, extend the sides of any block beyond the frame border, as shown below, to form a unit equilateral triangle and a smaller equilateral triangle of side length $1 - d$.

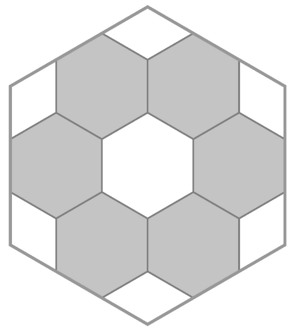


Each empty region along the border of the frame can be divided into a unit equilateral triangle and a trapezoid with bases of lengths 1 and $1-d$. Thus the frame has a side length of $(1-d)+1+1+d = 3$ units and encloses an area of $6 \cdot \frac{9\sqrt{3}}{4} = \frac{27\sqrt{3}}{2}$ square units. The required area is $\frac{27\sqrt{3}}{2} - 9\sqrt{3} = \frac{9\sqrt{3}}{2}$ square units. Note that the value of d is irrelevant.

OR

The required area equals the difference between the area enclosed by the hexagonal frame and the area of the 6 hexagonal blocks. A block is composed of 6 unit equilateral triangles, each of area $\frac{\sqrt{3}}{4}$ square units, so the 6 blocks have a total area of $6 \cdot 6 \cdot \frac{\sqrt{3}}{4} = 9\sqrt{3}$ square units.

To determine the side length of the hexagonal frame, note that the blocks can be shifted simultaneously without altering the size of the frame. Consider the arrangement shown below.

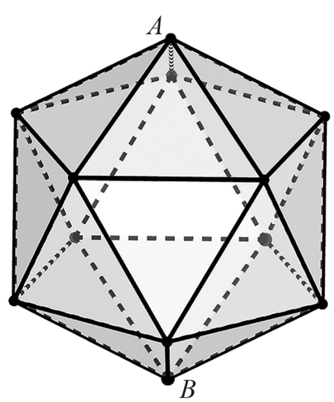


If the blocks are moved to the centers of the frame edges, there will be an empty 60° rhombus of side length 1 unit at each corner of the frame. It follows that the side length of the frame is $1 + 1 + 1 = 3$ units and encloses an area of $\frac{27\sqrt{3}}{2}$ square units. Thus the required area is $\frac{27\sqrt{3}}{2} - 9\sqrt{3} = \frac{9\sqrt{3}}{2}$ square units.

25. If A and B are vertices of a polyhedron, define the *distance* $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

(A) $\frac{7}{22}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{1}{2}$

Answer (A): One way to envision a regular icosahedron is with pentagonal pyramids on top and bottom, joined by a pentagonal antiprism. Let vertex A be at the top, as illustrated.



There are five vertices on the base of the pyramid with apex A, so they are all at distance 1 from A. There are five vertices on the base of the pyramid with apex B, which are all at distance 1 from the previous five vertices, so they are at distance 2 from A. Finally, only vertex B is at distance 3 from A. Thus from any vertex, there are 5 vertices at distance 1, 5 at distance 2, and 1 at distance 3. Randomly select Q, R, and S from the 12 vertices. Let $\mathbb{P}\big[E\big]$ denote the probability of event E, and let $\mathbb{P}\big[E \mid F\big]$ denote the probability of event E given event F. By symmetry

$$\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \mathbb{P}\big[d(Q,R) < d(R,S)\big],$$

so it suffices to compute $\mathbb{P}\big[d(Q,R) = d(R,S)\big]$ and divide its complement by 2 to find $\mathbb{P}\big[d(Q,R) > d(R,S)\big]$. Note that $\mathbb{P}\big[d(Q,R) = 1\big] = \frac{5}{11}$ and

$$\mathbb{P}\big[d(R,S) = 1 \mid d(Q,R) = 1\big] = \frac{4}{10} = \frac{2}{5},$$

because vertex Q is not available. Therefore

$$\mathbb{P}\big[d(Q,R) = d(R,S) = 1\big] = \frac{5}{11} \cdot \frac{2}{5} = \frac{2}{11}.$$

Because the number of vertices that are at distance 2 from Q is also 5, $\mathbb{P}\big[d(Q,R) = d(R,S) = 2\big]$ is also $\frac{2}{11}$.

Note that $d(Q,R) = d(R,S) = 3$ is impossible, because Q and S are distinct. Therefore

$$\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \frac{1}{2}\left(1 - \frac{2}{11} - \frac{2}{11}\right) = \frac{7}{22}.$$

OR

Use the same setting and notation as in the first solution. If $d(Q,R) = 1$, then $d(Q,R) \not> d(R,S)$. If $d(Q,R) = 3$, then $d(Q,R) > d(R,S)$, and this happens with probability $\frac{1}{11}$. If $d(Q,R) = 2$, then $d(Q,R) > d(R,S)$ exactly when $d(R,S) = 1$, which happens for 5 vertices of the remaining 10 from which to choose. Because $\mathbb{P}\big[d(Q,R) = 2\big] = \frac{5}{11}$,

$$\mathbb{P}\big[d(R,S) = 1 \mid d(Q,R) = 2\big] = \frac{5}{10} = \frac{1}{2},$$

so

$$\mathbb{P}\big[d(Q,R) > d(R,S) \text{ and } d(Q,R) = 2\big] = \frac{5}{11} \cdot \frac{1}{2} = \frac{5}{22}.$$

The two ways found are mutually exclusive, so $\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \frac{1}{11} + \frac{5}{22} = \frac{7}{22}$.

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